Inequalities

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The Arithmetic Mean – Geometric Mean (AM-GM) Inequality:

For any two positive real numbers a and b, we have

$$\frac{a+b}{2} \geq \sqrt{ab}$$

with equality if and only if $a = b$.

- The quantity of the LHS is called the *arithmetic mean* of the two numbers a and b.
- The quantity of the LHS is called the *geometric mean* of the two numbers a and b.
- They can be regarded as providing two different ways of "averaging" a pair of numbers.

Remark: This result has the following interpretations:

- The *minimum* value of the *sum* of two positive quantities whose *product* is fixed occurs when both are equal.
- The *maximum* value of the *product* of two positive quantities whose *sum* is fixed occurs when both are equal.
- A geometric interpretation of this result is that in any right-angled triangle, the median corresponding to the hypotenuse is bigger than the altitude corresponding to hypothenuse.

Example. Find the minimum of $4x + \frac{9}{x}$, where x is positive.

Solution. By the AM-GM inequality,

$$4x + \frac{9}{x} \geq 2\sqrt{(4x) \cdot \left(\frac{9}{x}\right)}$$
$$= 2\sqrt{36} = 12 .$$

The minimum occurs when $4x = \frac{9}{x}$, i.e., when $x = 3/2$.

Example. Prove that for any positive numbers a, b and c we have

$$(a+b)(b+c)(c+a) \geq 8abc.$$

Solution. By the AM-GM inequality we have

$$\frac{a+b}{2} \geq \sqrt{ab}, \quad \frac{b+c}{2} \geq \sqrt{bc}, \quad \frac{c+a}{2} \geq \sqrt{ca}$$

If we multiply these three inequalities we find

$$\frac{(a+b)(b+c)(c+a)}{8} \geq \sqrt{(ab)(bc)(ca)} = abc$$

and this finishes our proof.

The Arithmetic Mean – Geometric Mean (AM-GM) Inequality (more than two variables):

Suppose we have n positive real numbers $x_1, x_2, \ldots, x_n$. Then

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq (x_1 x_2 \cdots x_n)^{\frac{1}{n}}$$

with equality if and only if all of the numbers $x_1, x_2, \ldots, x_n$ are equal.

Remark: This result has the following interpretations:

- The *minimum* value of the *sum* of positive quantities whose *product* is fixed occurs when all are equal.
- The *maximum* value of the *product* of positive quantities whose *sum* is fixed occurs when all are equal.

Example.

Minimize $x^2 + y^2 + z^2$ subject to $x, y, z > 0$ and $xyz = 1$.

Solution. By AM-GM,

$$\begin{aligned} x^2 + y^2 + z^2 &\geq 3\sqrt[3]{x^2 \cdot y^2 \cdot z^2} \\ &= \sqrt[3]{(xyz)^2} \\ &= 1\ . \end{aligned}$$

The minimum occurs when $x^2 = y^2 = z^2$, i.e., when $x = y = z = 1$.

Example.

Minimize $\frac{6x}{y} + \frac{12y}{z} + \frac{3z}{x}$ for $x, y, z > 0$.

Solution. By AM-GM,

$$\frac{6x}{y} + \frac{12y}{z} + \frac{3z}{x} \geq 3\sqrt[3]{\frac{6x}{y} \cdot \frac{12y}{z} \cdot \frac{3z}{x}} = 3\sqrt[3]{6 \cdot 12 \cdot 3} = 3 \cdot 6 = 18\ .$$

The minimum occurs if and only if $\frac{6x}{y} = \frac{12y}{z} = \frac{3z}{x}$, i.e., if and only if $x = t$, $y = t$ and $z = 2t$ for some positive number t.

Example.

Maximize $xy\,(72 - 3x - 4y)$, where $x, y > 0$ and $3x + 4y < 72$.

Solution. We seek to maximize the product of three positive quantities. Note that the sum of the three quantities is equal to

$$x + y + (72 - 3x - 4y) = 72 - 2x - 3y\ .$$

This is NOT a constant! However, we can rearrange the product as

$$xy\,(72 - 3x - 4y) = \frac{1}{12}\,(3x)\,(4y)\,(72 - 3x - 4y)$$

Thus by AM-GM, the maximum occurs when $3x = 4y = 72 - 3x - 4y$, i.e., when $3x = 72 - 6x$. This yields $9x = 72$, or $x = 8$. Thus $y = 6$ and the maximum value is $\frac{1}{12} \cdot (24)^3 = 1152$.

Example.

Let a be a positive constant. Minimize $x^2 + \frac{a}{x}$, where $x > 0$.

Solution. We seek to minimize the sum of two quantities. Note that the product of the two quantities is equal to ax – this is NOT a constant. However, we can rearrange the sum as

$$x^2 + \frac{a}{2x} + \frac{a}{2x} \ .$$

Thus using AM-GM,

$$x^2 + \frac{a}{2x} + \frac{a}{2x} \geq 3\sqrt[3]{x^2 \cdot \frac{a}{2x} \cdot \frac{a}{2x}} = 3\sqrt[3]{\frac{a^2}{4}} = 3\left(\frac{a}{2}\right)^{\frac{3}{2}} \ .$$

The minimum occurs when $x^2 = \frac{a}{2x} = \frac{a}{2x}$, i.e. when $x = \sqrt[3]{\frac{a}{2}}$.

Two More "Averages":

The **Harmonic Mean** of n numbers $x_1, x_2, \ldots, x_n$ is given by

$$\mathrm{HM} = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$$

and their **Root-Mean-Square** is given by

$$\mathrm{RMS} = \sqrt{\frac{x_1^2 + x_2^2 + \cdots + x_n^2}{n}}\ .$$

If all the numbers $x_1, x_2, \ldots, x_n$ are positive, then we have

$$\min\{x_1, \ldots, x_n\} \leq \mathrm{HM} \leq \mathrm{GM} \leq \mathrm{AM} \leq \mathrm{RMS} \leq \max\{x_1, \ldots, x_n\}$$

with equality in each case if and only if all of the numbers $x_1, x_2, \ldots, x_n$ are equal.

Special case: for two positive numbers x and y

$$\min\{x, y\} \leq \frac{2xy}{x+y} \leq \sqrt{xy} \leq \frac{x+y}{2} \leq \sqrt{\frac{x^2+y^2}{2}} \leq \max\{x, y\}\ .$$

Exercise: Prove the above special case (*all* inequalities)!

Looking at the AM-HM inequality, we have AM $\geq$ HM, or

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}$$

This can be rearranged into the form

$$(x_1 + x_2 + \cdots + x_n)\left(\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}\right) \geq n^2 \, ,$$

with equality if and only if the numbers $x_1, x_2, \ldots, x_n$ are all equal.

Example: "Nesbitt's Inequality".

Prove that for positive numbers a, b, c,

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2} \, .$$

Solution. Write the LHS as

$$\begin{aligned}
&\frac{a+b+c}{b+c} + \frac{a+b+c}{a+c} + \frac{a+b+c}{a+b} - 3 \\
&\qquad = (a+b+c)\left(\frac{1}{b+c} + \frac{1}{a+c} + \frac{1}{a+b}\right) - 3 \\
&\qquad = \frac{1}{2}\left[(a+b) + (b+c) + (a+c)\right]\left[\frac{1}{b+c} + \frac{1}{a+c} + \frac{1}{a+b}\right] - 3 \\
&\qquad \geq \frac{1}{2}(9) - 3 = \frac{3}{2}
\end{aligned}$$

where we have used the HM-AM inequality with $n = 3$:

$$(x + y + z)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \geq 3^2$$

with $x = a + b$, $y = b + c$, $z = a + c$.

Sometimes we can be asked to prove an inequality regarding the *sides lengths of a triangle*. Here, the side lengths a, b, c (aside from being positive) must satisfy the so-called *triangle inequalities*:

$$a+b>c\,; \quad b+c>a\,; \quad c+a>b\,;$$

Example.

Let a, b, c be the side lengths of a triangle. Prove that

$$a^2+b^2+c^2<2(ab+bc+ca).$$

Solution.

Note that for example, if $a=5$ and $b=c=1$, we have

$$a^2+b^2+c^2=27\,; \quad 2(ab+bc+ca)=22.$$

and the result does *not* hold. Therefore, it is important that we use the information that a, b, c satisfy the triangle inequalities.

Writing the triangle inequality $a+b>c$ as $c-b<a$ and squaring, we obtain $(c-b)^2<a^2$. Doing this for each triangle inequality yields

$$(c-b)^2<a^2$$

$$(a-b)^2<c^2$$

$$(c-a)^2<b^2$$

Adding these three inequalities, and simplifying, yields the result (**Exercise:** check this!).

Example. (Variation of an Irish Math Olympiad 2016 problem) Let $n \geq 1$ and $a_1, a_2, \ldots, a_{2n}$ be positive integers, none of which is equal to 10, such that

$$a_1 + a_2 + \cdots + a_{2n} = 20n.$$

Prove that

$$a_1 a_2 \cdots a_{2n} \leq 99^n.$$

Solution. Rearranging the numbers $a_1, a_2, \ldots, a_{2n}$ we may asume

$$a_1 \leq a_2 \leq \cdots \leq a_{2n}.$$

Denote

$$A = \frac{a_1 + a_2 + \cdots + a_n}{n}, \quad B = \frac{a_{n+1} + a_{n+2} + \cdots + a_{2n}}{n}.$$

Then $A \leq B$ and $A + B = 20$. Thus, $A = 10 - x$, $B = 10 + x$ for some $x \geq 0$.

We claim that $x \geq 1$. Indeed, assuming $x < 1$ it follows that $A = 10 - x > 9$. This shows that $a_n > 9$ so

$$a_{2n} \geq a_{2n-1} \geq \cdots \geq a_{n+1} \geq a_n > 9$$

and because none of these numbers is equal to 10, it follows that

$$a_{2n} \geq a_{2n-1} \geq \cdots \geq a_{n+1} \geq 11$$

and hence $B \geq 11$. But this contradicts $A + B = 20$. Hence $x \geq 1$.

By AM-GM inequality we have

$$(a_1 a_2 \ldots a_n)^{1/n} \leq A \quad \text{and} \quad (a_{n+1} a_{n+2} \ldots a_{2n})^{1/n} \leq B$$

Multiplying these inequalities we arrive at

$$(a_1 a_2 \cdots a_{2n})^{1/n} \leq AB = (10 - x)(10 + x) = 100 - x^2 \leq 99$$

which yields

$$a_1 a_2 \cdots a_{2n} \leq 99^n.$$

Note that equality may occur for instance for $a_1 = a_2 = \cdots = a_n = 9$ ands $a_{n+1} = a_{n+2} = \cdots = a_{2n} = 11$.

Exercises.

(1) Let $x, y > 0$. Find the minimum of

$$\frac{50}{x} + \frac{20}{y} + xy.$$

(2) If $x > y > 0$, find the minimum of

$$x + \frac{8}{y(x-y)}.$$

(3) Prove that for any positive real numbers a, b, c we have

$$(a + 9b)(b + 9c)(c + 9a) \geq 216abc.$$

(4) Prove that for positive real numbers x, y, z,

$$x^2 + y^2 + z^2 \geq xy + yz + zx,$$

and determine when equality occurs.

(5) Find the positive number whose square exceeds its cube by the greatest amount.

(6) Prove that for positive real numbers x, y, z,

$$\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \leq \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right).$$

(7) The sum of a number of positive integers is 2017. Determine the maximum value their product could have.

For further reading, click here: Wikipedia entry on AM-GM